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Two-Dimensional $\mathbf{N = (2, 2)}$ Dilaton Supergravity from Graded Poisson-Sigma Models II:

Analytic Solution and BPS States

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Abstract

The integrability of $N = (2, 2)$ dilaton supergravity in two dimensions is studied by the use of the graded Poisson Sigma model approach. Though important differences compared to the purely bosonic models are found, the general analytic solutions are obtained. The latter include minimally gauged models as well as an ungauged version. BPS solutions are an especially interesting subclass.

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1 Introduction

Graded Poisson Sigma (gPSM) models [1–6] have proved to represent a powerful formalism in the quest to solve several longstanding problems in 2d (dilaton) supergravity [7,8]. As shown in our first paper [9] on the subject of $N = (2, 2)$ supergravity [10–22] it is possible to formulate the full actions including all fermionic contributions in a compact form. Although our version works with a non-linear and open algebra, this turns out to be no disadvantage thanks to the powerful symmetry principles of the gPSM. In this way the very complicated and lengthy actions that follow from superspace at the component level (cf. e.g. [15–18]) can be avoided. It turns out that the existence of the equivalent gPSM formulation is the key ingredient for the numerous successes of this method which even extends to the quantization of such theories [23, 24]. On the other hand, there is an isomporphic mapping of the symmetries as given by the superfield formulation for $N = (2, 2)$ onto the ones in the field content of gPSMs [9].

Our present paper is an immediate continuation of this work by using another convenient feature of the gPSM formalism, namely the possibility to derive the full classical solution, including the complete fermionic contributions. Though many aspects of the calculation are a rather straightforward generalization of previous results, new important problems appear which are related to the question of the existence of Casimir-Darboux coordinates on graded Poisson manifolds.

In order to provide a sufficiently self-contained presentation we again start (Section 2) with a condensed description of the gPSM, summarizing also the main results of [9] as needed for the application to $N = (2, 2)$ supergravity in the present work. Section 3 contains the solution for the chiral version of $N = (2, 2)$ dilaton supergravity. It is enough to consider the case which corresponds to vanishing kinetic term of the dilaton field in the version formulated as a dilaton theory, because the general case can be obtained by straightforward conformal transformation (Section 4). The twisted chiral case is covered as well by a simple “mirror-type” redefinition of fields. Section 5 is devoted to a formulation of ungauged $N = (2, 2)$ supergravity which consists in restricting the previous theory to a fixed leaf of one of the Casimir functions in a gauge theory. A short discussion of BPS solutions is the subject of Section 6, where we show that (even in the so much simpler gPSM approach) the complications for $N = (2, 2)$ as compared to $N = (1, 1)$ supergravity [25] at present allow a consideration of the bosonic part only. After the conclusion (Section 7) we decided to include as in [9] again the Appendix describing the notation somewhat more in detail.

2 gPSM for $\mathbf{N = (2, 2)}$ supergravity

In this Section some relevant formulae of (graded) Poisson Sigma models and their application in dilaton supergravity are reviewed. For further details ref. [9] and earlier

literature on the topic, esp. [4, 6, 26, 27] should be consulted. A general gPSM consists of scalar fields $X^I(x)$, which are themselves coordinates of a graded Poisson manifold with Poisson tensor $P^{IJ}(X) = (-1)^{IJ+1}P^{JI}(X)$. The index I, in the generic case, includes commuting as well as anti-commuting fields[1]. In addition one introduces the gauge potential $A = \mathrm{d}X^I A_I = \mathrm{d}X^I A_{mI}(x)\, \mathrm{d}x^m$, a one form with respect to the Poisson structure as well as with respect to the 2d worldsheet coordinates. The gPSM action reads[2]

$$\mathcal{S}_{gPSM} = \int_M \mathrm{d}X^I \wedge A_I + \frac{1}{2}P^{IJ} A_J \wedge A_I \ . \tag{2.1}$$

The Poisson tensor P^{IJ} must have vanishing Nijenhuis tensor (obey a Jacobi-type identity with respect to the Schouten bracket related as $\{X^I, X^J\} = P^{IJ}$ to the Poisson tensor)

$$J^{IJK} = P^{IL}\partial_L P^{JK} + g\text{-}perm\,(IJK) = 0 \ , \tag{2.2}$$

where the sum runs over the graded permutations. The variation of A_I and X^I in (2.1) yields the gPSM equations of motion (eom-s)

$$\mathrm{d}X^I + P^{IJ} A_J = 0 \ , \tag{2.3}$$

$$\mathrm{d}A_I + \frac{1}{2}(\partial_I P^{JK}) A_K A_J = 0 \ . \tag{2.4}$$

Due to (2.2) the action (2.1) is invariant under the symmetry transformations

$$\delta X^I = P^{IJ}\varepsilon_J \ , \qquad \delta A_I = -\mathrm{d}\varepsilon_I - \left(\partial_I P^{JK}\right)\varepsilon_K\, A_J \ , \tag{2.5}$$

where the term $\mathrm{d}\epsilon_I$ in the second of these equations provides the justification for calling A_I "gauge fields".

If the Poisson tensor has a non-vanishing kernel there exist (one or more) Casimir functions $C(X)$ obeying

$$\{X^I, C\} = P^{IJ}\frac{\partial C}{\partial X^J} = 0 \ , \tag{2.6}$$

which, when the X^I obey the field equations of motion, are constants of motion.

It was shown in ref. [9] that minimally gauged $N = (2, 2)$ dilaton supergravity can be described by a gPSM if the target space has four (real) commuting dimensions, interpreted as a complex dilaton $X = \phi + i\pi$ and an auxiliary field X^a, and four anti-commuting ones, which are combined in a complex two-component dilatino χ^α. The associated gauge fields are the spin-connection ω, the "zweibein" e_a and the complex

[1] The usage of different indices as well as other features of our notation are explained in Appendix A. For further details one should consult ref. [4, 28].

[2] If the multiplication of forms is evident in what follows, the wedge symbol will be omitted.

gravitino ψ_α. For a gauged $U(1)$ symmetry of χ^α another $U(1)$ gauge field B must be introduced. General dilaton supergravity models are completely determined by two X-dependent functions, namely a prepotential $u(X, \bar{X})$ and the conformal factor $Q(X)$. The derivative of the latter is denoted as $Q'(X) = Z(X)$ and defines the contributions quadratic in bosonic torsion. Furthermore it is useful to introduce the potentials w and W, which control the bosonic and fermionic parts, respectively:

$$w(X) = \frac{1}{4} e^{\bar{Q}/2} u \qquad\qquad W(X, \bar{X}) = -2w\bar{w} \tag{2.7}$$

With these definitions general chiral dilaton supergravity is described by the Poisson tensor [9]

$$P^{a\phi} = X^b \epsilon_b{}^a \ , \qquad P^{\pi\phi} = 0 \ , \qquad P^{\alpha\phi} = -\frac{1}{2}\chi^\beta \gamma_{*\beta}{}^\alpha \ , \qquad P^{\bar{\alpha}\phi} = -\frac{1}{2}\bar{\chi}^\beta \gamma_{*\beta}{}^\alpha \ . \tag{2.8}$$

$$P^{a\pi} = 0, \qquad P^{\alpha\pi} = -\frac{i}{2}\chi^\beta \gamma_{*\beta}{}^\alpha \ , \qquad P^{\bar{\alpha}\pi} = \frac{i}{2}\bar{\chi}^\beta \gamma_{*\beta}{}^\alpha \ . \tag{2.9}$$

$$P^{ab} = \epsilon^{ab}\Big(e^{-(Q+\bar{Q})/2} W' + \frac{1}{2} Y (Z + \bar{Z}) + \frac{1}{4}\chi^2 e^{-Q/2} \bar{w}'' + \frac{1}{4}\bar{\chi}^2 e^{-\bar{Q}/2} w'' \Big) \ , \tag{2.10}$$

$$P^{a\alpha} = i e^{-\bar{Q}/2} w' (\bar{\chi}\gamma^a)^\alpha - \frac{\bar{Z}}{4} X^b (\chi \gamma_b \gamma^a \gamma_*)^\alpha \ , \tag{2.11}$$

$$P^{a\bar{\alpha}} = i e^{-Q/2} \bar{w}' (\chi\gamma^a)^\alpha - \frac{Z}{4} X^b (\bar{\chi} \gamma_b \gamma^a \gamma_*)^\alpha \ , \tag{2.12}$$

$$P^{\alpha\bar{\beta}} = -2i X^a (\gamma_a)^{\alpha\beta} \ , \tag{2.13}$$

$$P^{\alpha\beta} = (u + \frac{\bar{Z}}{4}\chi^2)\gamma_*{}^{\alpha\beta} \ , \qquad\qquad P^{\bar{\alpha}\bar{\beta}} = (\bar{u} + \frac{Z}{4}\bar{\chi}^2)\gamma_*{}^{\alpha\beta} \ . \tag{2.14}$$

The bosonic part of the Poisson tensor has four dimensions but at most rank two. Therefore there exist at least two (real) commuting Casimir functions, which can be chosen as

$$C = 8\big(W + e^{(Q+\bar{Q})/2} (Y + \frac{1}{4}\chi^2 e^{-Q/2} \bar{w}' + \frac{1}{4}\bar{\chi}^2 e^{-\bar{Q}/2} w') \big) \ , \tag{2.15}$$

$$C_\pi = \pi + i e^{\bar{Q}/2} \frac{\bar{w}}{C} \chi^2 - e^{Q/2} \frac{w}{C} \bar{\chi}^2 - \frac{e^{(Q+\bar{Q})/2}}{C} X^a (\chi \gamma_a \gamma_* \bar{\chi}) \ . \tag{2.16}$$

The first one is related to the energy[3] the second to the $U(1)$ charge of the system.

An important simplified model is dilaton prepotential supergravity [4] obtained for the special case $Z = 0$ (cf. Section 3 of [9]). General supergravity models can be obtained from the latter by means of conformal transformations, which are interpreted as

[3]This energy conservation is a pecular feature of 2d (super-)gravity, even in the presence of matter [25, 29, 30].

target-space diffeomorphisms. Therefore, for any *local* analysis it is sufficient as a first step to consider this simpler class of models. Nevertheless, the conformal transformations towards $Z \neq 0$ need not be defined globally and thus the latter models tend to be physically inequivalent.

Finally we note that under the exchange $\chi^- \leftrightarrow \bar{\chi}^-$ and $\psi_- \leftrightarrow \bar{\psi}_-$ a chiral gauging of the internal $U(1)$ turns into a twisted chiral one. This map represents mirror symmetry; it is defined globally and thus physics do not change, as is expected for the geometric part of the action.

3 Solution of dilaton prepotential SUGRA

The aim of this work is to study the integrability of $N = (2,2)$ dilaton supergravity and to derive its analytic solution. As announced above the explicit calculations can be restricted to the chiral version of dilaton prepotential supergravity, the general theories are then obtained by means of conformal transformations. Although we find agreement with the general statements about (g)PSMs that the models developed in this work are integrable, important differences between graded PSMs and ordinary (bosonic) PSMs become manifest here.

3.1 gPSM and Casimir-Darboux coordinates

The integrability of bosonic dilaton gravity may be checked by explicit integration of the equations of motion [29]. However, once the theory is formulated in terms of a PSM this characteristic is guaranteed by the fact that any Poisson manifold locally can be transformed to Casimir-Darboux (CD) coordinates[4]. As the integrability of the model at hand is not obvious in their "physical" coordinates, it is helpful to choose new coordinates that are CD or at least almost CD.

In the following we assume that $X^{++} \neq 0$. Then for the purely bosonic theory one can choose the Casimir-Darboux coordinates $\{C, \pi, \phi, \lambda\}$ with $\lambda = -\ln|X^{++}|$. The only non-vanishing bracket among these variables is $\{\lambda, \phi\} = 1$. As π does not commute with the fermions, this choice does not lead to CD coordinates for the full theory, but they would be $\{C, C_\pi, \phi, \lambda\}$ plus some convenient choice for the fermions. However the former choice turns out to be the preferrable one: First, the replacement $\pi \to C_\pi$ leads to lengthy equations (cf. Section 7 of [9]) and second, solutions for $C_B = 0$ cannot be obtained in this way, as C_π contains inverse powers in this function.

[4] Such coordinates exist on regular sheets of the Poisson manifold, only. Solutions on irregular sheets have to be considered seperately [31, 32]. In the bosonic model these are restricted to the point $X^{++} = X^{--} = 0$. Such solutions describe constant dilaton vacua or a bifurcation point. Some solutions of this type are discussed in Section 6.

Among the fermionic coordinates we follow the idea of ref. [4] to choose the Lorentz invariant quantities[5]

$$\tilde{\chi}^{(+)} = \frac{1}{\sqrt{|X^{++}|}}\chi^+ \ , \qquad \tilde{\bar{\chi}}^{(+)} = \frac{1}{\sqrt{|X^{++}|}}\bar{\chi}^+ \ , \tag{3.1}$$

$$\hat{\chi}^{(-)} = \sqrt{|X^{++}|}\chi^- - \frac{i\sigma u}{2\sqrt{2}}\tilde{\bar{\chi}}^{(+)} \ , \qquad \hat{\bar{\chi}}^{(-)} = \sqrt{|X^{++}|}\bar{\chi}^- - \frac{i\sigma \bar{u}}{2\sqrt{2}}\tilde{\chi}^{(+)} \ , \tag{3.2}$$

as new coordinates (σ denotes the sign[6] of iX^{++}). The second term in the definition of $\hat{\chi}^{(-)}$ is motivated by the bracket

$$\{\lambda, \sqrt{|X^{++}|}\chi^-\} = \{\lambda, \tilde{\chi}^{(-)}\} = \frac{i\sigma}{2\sqrt{2}}u'\tilde{\bar{\chi}}^{(+)} \ . \tag{3.3}$$

It is now straightforward to check that the Poisson brackets—beside the purely bosonic ones already mentioned above—reduce to

$$\{\pi, \tilde{\chi}^{(+)}\} = \frac{i}{2}\tilde{\chi}^{(+)} \ , \qquad \{\pi, \tilde{\bar{\chi}}^{(+)}\} = -\frac{i}{2}\tilde{\bar{\chi}}^{(+)} \ , \tag{3.4}$$

$$\{\pi, \hat{\chi}^{(-)}\} = -\frac{i}{2}\hat{\chi}^{(-)} \ , \qquad \{\pi, \hat{\bar{\chi}}^{(-)}\} = \frac{i}{2}\hat{\bar{\chi}}^{(-)} \ , \tag{3.5}$$

$$\{\tilde{\chi}^{(+)}, \tilde{\bar{\chi}}^{(+)}\} = -2\sqrt{2}i\sigma \ , \qquad \{\hat{\chi}^{(-)}, \hat{\bar{\chi}}^{(-)}\} = -\frac{i\sigma}{2\sqrt{2}}C \ , \tag{3.6}$$

while all remaining brackets are zero. All details of the model are hidden in the redefinition of the fields, and the equations of motion for the new variables become independent of the prepotential u. To distinguish the set of transformed gauge potentials from the original ones they all are denoted with a tilde ($\tilde{A}_C, \tilde{A}_\pi, \tilde{A}_\lambda, \tilde{A}_\phi, \tilde{A}_{(+)}, \tilde{\bar{A}}_{(+)}, \tilde{A}_{(-)}, \tilde{\bar{A}}_{(-)}$). Also, the action (2.1) is expressed in terms of the transformed Poisson tensor related to the brackets (3.4)-(3.6) and $\{\lambda, \phi\} = 1$. Variation of the action with respect to these

[5]Throughout Section 3 variables like $\chi^\pm$ etc. refer to the restricted case $Z = 0$. In the transition to $Z \neq 0$ in Section 4 we shall rename the variables of Section 2 by underlining them.

[6]According to the conventions outlined in the Appendix, quantities like $X^{\pm\pm}$ are imaginary (cf. eqs. (A.10) and (A.11)). In refs. [4,6] a real value for X^{++} has been assumed, which corresponded to a slightly different convention.

$\tilde{A}_I$ yields the eom-s

$$\mathrm{d}C = 0 \ , \tag{3.7}$$

$$\mathrm{d}\pi + \frac{i}{2}(\tilde{\chi}^{(+)}\tilde{A}_{(+)} - \tilde{\tilde{\chi}}^{(+)}\tilde{\tilde{A}}_{(+)} - \hat{\chi}^{(-)}\tilde{A}_{(-)} + \hat{\hat{\chi}}^{(-)}\tilde{\tilde{A}}_{(-)}) = 0 \ , \tag{3.8}$$

$$\mathrm{d}\phi - \tilde{A}_\lambda = 0 \ , \qquad \mathrm{d}\lambda + \tilde{A}_\phi = 0 \ , \tag{3.9}$$

$$\mathrm{d}\tilde{\chi}^{(+)} + 2\sqrt{2}i\sigma\tilde{\tilde{A}}_{(+)} - \frac{i}{2}\tilde{\chi}^{(+)}\tilde{A}_\pi = 0 \ , \qquad \mathrm{d}\tilde{\tilde{\chi}}^{(+)} + 2\sqrt{2}i\sigma\tilde{A}_{(+)} + \frac{i}{2}\tilde{\tilde{\chi}}^{(+)}\tilde{A}_\pi = 0 \ , \tag{3.10}$$

$$\mathrm{d}\hat{\chi}^{(-)} - \frac{i\sigma}{2\sqrt{2}}C\tilde{\tilde{A}}_{(-)} + \frac{i}{2}\hat{\chi}^{(-)}\tilde{A}_\pi = 0 \ , \qquad \mathrm{d}\hat{\hat{\chi}}^{(-)} - \frac{i\sigma}{2\sqrt{2}}C\tilde{A}_{(-)} - \frac{i}{2}\hat{\hat{\chi}}^{(-)}\tilde{A}_\pi = 0 \ , \tag{3.11}$$

while variation with respect to X^I produces

$$\mathrm{d}\tilde{A}_C - \frac{i\sigma}{2\sqrt{2}}\tilde{A}_{(-)} \wedge \tilde{\tilde{A}}_{(-)} = 0 \ , \tag{3.12}$$

$$\mathrm{d}\tilde{A}_\pi = 0 \ , \qquad \mathrm{d}\tilde{A}_\phi = 0 \qquad \mathrm{d}\tilde{A}_\lambda = 0 \ , \tag{3.13}$$

$$\mathrm{d}\tilde{A}_{(+)} + \frac{i}{2}\tilde{A}_{(+)} \wedge \tilde{A}_\pi = 0 \ , \qquad \mathrm{d}\tilde{\tilde{A}}_{(+)} - \frac{i}{2}\tilde{\tilde{A}}_{(+)} \wedge \tilde{A}_\pi = 0 \ , \tag{3.14}$$

$$\mathrm{d}\tilde{A}_{(-)} - \frac{i}{2}\tilde{A}_{(-)} \wedge \tilde{A}_\pi = 0 \ , \qquad \mathrm{d}\tilde{\tilde{A}}_{(-)} + \frac{i}{2}\tilde{\tilde{A}}_{(-)} \wedge \tilde{A}_\pi = 0 \ . \tag{3.15}$$

It is obvious from the definition of C_π in (2.16) and the second equation in (3.6) that complications arise if the body of the Casimir function C vanishes. A similar problem already appears in $N = (1,1)$ supergravity [4] and—as a concise discussion of this point has not been given so far—this simpler model is considered first. As is easily seen from the above result by reducing configuration space to the one of $N = (1,1)$, CD coordinates are obtained after simple rescalings by negative powers of $\sqrt{C}$ except for the bracket in (3.6) (cf. Section 8 in [4], unimportant constants and factors of i are omitted)

$$\{\hat{\chi}^{(-)}, \hat{\chi}^{(-)}\} = C \tag{3.16}$$

and the respective eom-s

$$\mathrm{d}\hat{\chi}^{(-)} - C\tilde{A}_{(-)} = 0 \ , \qquad\qquad \mathrm{d}\tilde{A}_{(-)} = 0 \ . \tag{3.17}$$

Clearly (3.17) can be integrated for any value of $C = \text{const.}$ $\tilde{A}_{(-)}$ is closed and thus locally $\tilde{A}_{(-)} = \mathrm{d}\zeta_{(-)}$. For constant C the simple equation $\hat{\chi}^{(-)} = C\zeta_{(-)} + \hat{\chi}_0^{(-)}$ is obtained with arbitrary $\zeta_{(-)}$ and a constant $\hat{\chi}_0^{(-)}$. Nevertheless one has to distinguish three different cases:

1. $C = 0$. In that case the first equation in (3.17) defines $\hat{\chi}^{(-)}$ as a second anticommuting Casimir function $\hat{\chi}_0^{(-)}$ [4], $\tilde{A}_{(-)}$ is the associated gauge potential.

2. $C \neq 0$. This case is divided into two sub-classes:

   (a) Non-vanishing body of C. Then after a rescaling of $\hat{\chi}^{-}$ by $1/\sqrt{C}$ in (3.16) CD coordinates are obtained with C being the only Casimir function. Obviously the solution for $\tilde{A}_{(-)}$ can be expressed in terms of $\hat{\chi}^{(-)}$.

   (b) Vanishing body of C. This is a subtle case having no counterpart in the purely bosonic model. On the one hand, $\hat{\chi}^{(-)}$ is not a Casimir function as for $C = 0$, but on the other hand a division by C is excluded as C^{-1} does not exist. Therefore it is impossible to transform the system to CD coordinates and to express the solution for $\tilde{A}_{(-)}$ in terms of the target-space coordinates. From the first equation in (3.3) it even seems that the solution for $\hat{\chi}^{(-)}$ depends on $\zeta_{(-)}$. For the special case at hand it can be argued on general grounds that this is not the case [25]: As the solutions are parametrized by only two anti-commuting variables, $C\tilde{A}_{(-)}$ must vanish if C is pure soul. It is important to notice that this is a fortunate accident in this case.

3.2 Integration for generic Casimir function

We now show that a general solution can be obtained with the only restriction $X^{++} \neq 0$ (or equivalently $X^{--} \neq 0$.) The solutions of (3.9) and (3.13) are immediate with $\tilde{A}_\lambda = \mathrm{d}\phi$, $\tilde{A}_\phi = -\,\mathrm{d}\lambda$ and $\tilde{A}_\pi = -\,\mathrm{d}F_\pi$, where F_π is a free function. If C has non-vanishing body we could solve all four equations in (3.10)-(3.11) for the fermionic gauge potentials. Nevertheless we want to proceed in a different way, because solutions with $C = 0$ or $C =$ pure soul do appear in this model and in particular represent the interesting class of BPS states (cf. [25] and Section 6 below). From (3.10) we obtain

$$\tilde{A}_{(+)} = \frac{i\sigma}{2\sqrt{2}}\,\mathrm{d}\tilde{\bar{\chi}}^{(+)} + \frac{\sigma}{4\sqrt{2}}\tilde{\bar{\chi}}^{(+)}\,\mathrm{d}F_\pi\ , \qquad \tilde{\bar{A}}_{(+)} = \frac{i\sigma}{2\sqrt{2}}\,\mathrm{d}\tilde{\chi}^{(+)} - \frac{\sigma}{4\sqrt{2}}\tilde{\chi}^{(+)}\,\mathrm{d}F_\pi\ . \tag{3.18}$$

For the remaining fermionic variables the equations in (3.15) must be addressed first. They imply

$$\tilde{A}_{(-)} = \mathrm{d}\zeta_{(-)} + \frac{i}{2}\zeta_{(-)}\,\mathrm{d}F_\pi\ , \qquad \tilde{\bar{A}}_{(-)} = \mathrm{d}\bar{\zeta}_{(-)} - \frac{i}{2}\bar{\zeta}_{(-)}\,\mathrm{d}F_\pi \tag{3.19}$$

for some complex anti-commuting function $\zeta_{(-)}$. Now these solutions are inserted into the equations in (3.11) that yield after integration

$$\hat{\chi}^{(-)} = \frac{i\sigma}{2\sqrt{2}}C\bar{\zeta}_{(-)} + e^{\frac{i}{2}F_\pi}\lambda_0^{(-)}\ , \qquad \hat{\bar{\chi}}^{(-)} = \frac{i\sigma}{2\sqrt{2}}C\zeta_{(-)} + e^{-\frac{i}{2}F_\pi}\bar{\lambda}_0^{(-)}\ . \tag{3.20}$$

In this solution $\lambda_0^{(-)}$ is a constant spinor. Among the variations with respect to $\tilde{A}_I$ there remains (3.8), which should produce by $\mathrm{d}C_\pi = 0$ the constant of motion. When

(3.18) and (3.20) are inserted into that equation indeed a total derivative is obtained. Its integration with some integration constant C_π^0 yields[7]

$$C_\pi = \pi - \frac{\sigma}{4\sqrt{2}}\tilde{\chi}^{(+)}\tilde{\bar{\chi}}^{(+)} - \frac{\sigma}{4\sqrt{2}}C\zeta_{(-)}\bar{\zeta}_{(-)} - \frac{i}{2}(e^{\frac{i}{2}F_\pi}\lambda_0^{(-)}\zeta_{(-)} - e^{-\frac{i}{2}F_\pi}\bar{\lambda}_0^{(-)}\bar{\zeta}_{(-)}) + C_\pi^0 \,. \quad (3.21)$$

It remains to find the explicit form of $\tilde{A}_C$ from (3.12) and (3.19). This gauge potential depends on an additional free function $\mathrm{d}F$ and after a straightforward integration can be written as

$$\tilde{A}_C = -\,\mathrm{d}F + \frac{i\sigma}{4\sqrt{2}}\big(i\zeta_{(-)}\bar{\zeta}_{(-)}\,\mathrm{d}F_\pi - (\zeta_{(-)}\,\mathrm{d}\bar{\zeta}_{(-)} + \bar{\zeta}_{(-)}\,\mathrm{d}\zeta_{(-)})\big) \,. \quad (3.22)$$

Before proceeding to the discussion of specific classes of solutions we should worry about the transformations back to the original "physical" coordinates. The one of the gauge potentials follows straightforwardly by applying target space diffeomorphisms: $A_I = \partial\tilde{X}^J/\partial X^I \tilde{A}_J$. The $\tilde{X}^I$ comprise the CD coordinates of the bosonic sector as defined in the second paragraph of Section 3.1 together with the fermionic components (3.1) and (3.2). The explicit result reads:

$$\omega = \frac{\mathrm{d}X^{++}}{X^{++}} + \big(-(\bar{u}u)' + \chi^-\chi^+\bar{u}'' + \bar{\chi}^-\bar{\chi}^+u''\big)\tilde{A}_C - \frac{i\sigma u'}{2\sqrt{2}}\tilde{\bar{\chi}}^{(+)}\tilde{A}_{(-)} - \frac{i\sigma\bar{u}'}{2\sqrt{2}}\tilde{\chi}^{(+)}\tilde{\bar{A}}_{(-)} \quad (3.23)$$

$$B = -\,\mathrm{d}F_\pi - i\big((u'\bar{u} - u\bar{u}') + \chi^-\chi^+\bar{u}'' - \bar{\chi}^-\bar{\chi}^+u''\big)\tilde{A}_C + \frac{\sigma u'}{2\sqrt{2}}\tilde{\bar{\chi}}^{(+)}\tilde{A}_{(-)} - \frac{\sigma\bar{u}'}{2\sqrt{2}}\tilde{\chi}^{(+)}\tilde{\bar{A}}_{(-)} \quad (3.24)$$

$$\begin{aligned} e_{++} = &-\frac{\mathrm{d}\phi}{X^{++}} + 8X^{--}\tilde{A}_C - \frac{1}{2X^{++}}\big(\tilde{\chi}^{(+)}\tilde{A}_{(+)} + \tilde{\bar{\chi}}^{(+)}\tilde{\bar{A}}_{(+)} \\ &- (\tilde{\chi}^{(-)} + \frac{i\sigma u}{2\sqrt{2}}\tilde{\bar{\chi}}^{(+)})\tilde{A}_{(-)} - (\tilde{\bar{\chi}}^{(-)} + \frac{i\sigma\bar{u}}{2\sqrt{2}}\tilde{\chi}^{(+)})\tilde{\bar{A}}_{(-)}\big) \end{aligned} \quad (3.25)$$

$$e_{--} = 8X^{++}\tilde{A}_C \quad (3.26)$$

$$\psi_+ = \frac{1}{\sqrt{|X^{++}|}}\big(\tilde{A}_{(+)} - \frac{i\sigma\bar{u}}{2\sqrt{2}}\tilde{\bar{A}}_{(-)}\big) - \bar{u}'\chi^-\tilde{A}_C \quad (3.27)$$

$$\bar{\psi}_+ = \frac{1}{\sqrt{|X^{++}|}}\big(\tilde{\bar{A}}_{(+)} - \frac{i\sigma u}{2\sqrt{2}}\tilde{A}_{(-)}\big) - u'\bar{\chi}^-\tilde{A}_C \quad (3.28)$$

$$\psi_- = \sqrt{|X^{++}|}\tilde{A}_{(-)} + \bar{u}'\chi^+\tilde{A}_C \quad (3.29)$$

$$\bar{\psi}_- = \sqrt{|X^{++}|}\tilde{\bar{A}}_{(-)} + u'\bar{\chi}^+\tilde{A}_C \quad (3.30)$$

[7] Anticipating the result of Section 3.3 the constant C_π^0 cannot be set to zero in order to match the prescription in (2.16).

The dependence on a specific model is determined by the prepotential $u(\phi)$ alone.

The similarity of this solution to the one of bosonic gravity [27] as well as of $N = (1,1)$ supergravity [4, 6] is immediate. In the following we want to discuss more in detail the solution of dilaton prepotential supergravity for different values of the Casimir function C. The solution of the general supergravity model (2.7)-(2.16) with $Z \neq 0$ then simply follows by applying certain target space diffeomorphisms onto these solutions.

3.3 Non-vanishing body of the Casimir C

The simplest solution is obtained on a patch with non-vanishing body of the Casimir function, as in that case C^{-1} is well defined. Then also the definition of the second Casimir (2.16), taken at $Q = 0$, makes sense.

Thus we can solve still the equations in (3.20) for $\bar{\zeta}_{(-)}$ and $\zeta_{(-)}$, resp.:

$$\zeta_{(-)} = -\frac{2\sqrt{2}i\sigma}{C}(\hat{\bar{\chi}}^{(-)} - e^{-\frac{i}{2}F_\pi}\bar{\lambda}_0^{(-)}) \qquad \bar{\zeta}_{(-)} = -\frac{2\sqrt{2}i\sigma}{C}(\hat{\chi}^{(-)} - e^{\frac{i}{2}F_\pi}\lambda_0^{(-)}) \tag{3.31}$$

It is seen that the $\lambda_0^{(-)}$ part drops out of the solution for $\tilde{A}_{(-)}$:

$$\tilde{A}_{(-)} = -\frac{2\sqrt{2}i\sigma}{C}(\mathrm{d}\hat{\bar{\chi}}^{(-)} + \frac{i}{2}\hat{\bar{\chi}}^{(-)}\,\mathrm{d}F_\pi) \quad \tilde{\bar{A}}_{(-)} = -\frac{2\sqrt{2}i\sigma}{C}(\mathrm{d}\hat{\chi}^{(-)} - \frac{i}{2}\hat{\chi}^{(-)}\,\mathrm{d}F_\pi) \tag{3.32}$$

Of course, one could proceed by integrating again (3.8) and (3.12) with these formulas. Instead we insert (3.31) into the expressions (3.21) and (3.22) which leads to

$$C_\pi = \pi - \frac{\sigma}{4\sqrt{2}}\tilde{\chi}^{(+)}\tilde{\bar{\chi}}^{(+)} - \frac{\sqrt{2}\sigma}{C}(\hat{\chi}^{(-)}\hat{\bar{\chi}}^{(-)} - \lambda_0^{(-)}\bar{\lambda}_0^{(-)}) \; , \tag{3.33}$$

$$\begin{aligned} \tilde{A}_C = &-\mathrm{d}\big(F + \frac{\sqrt{2}i\sigma}{C^2}(e^{\frac{i}{2}F_\pi}\lambda_0^{(-)}\hat{\bar{\chi}}^{(-)} + e^{-\frac{i}{2}F_\pi}\bar{\lambda}_0^{(-)}\hat{\chi}^{(-)})\big) \\ &- \frac{\sqrt{2}\sigma}{C^2}\big(\hat{\chi}^{(-)}\hat{\bar{\chi}}^{(-)}\,\mathrm{d}F_\pi + i(\hat{\bar{\chi}}^{(-)}\,\mathrm{d}\hat{\chi}^{(-)} + \hat{\chi}^{(-)}\,\mathrm{d}\hat{\bar{\chi}}^{(-)})\big) \; . \end{aligned} \tag{3.34}$$

Starting instead with the definitions (3.32) one obtains the same result up to the terms dependent on $\lambda_0^{(-)}$, which are clearly absent in that case. However, in (3.33) the last term simply produces the constant C_π^0 in (3.21). In (3.34) the terms $\propto \lambda_0^{(-)}$ can be absorbed by a redefinition of F.

To summarize: this solution is parametrized by the two Casimir functions according to (2.15) and (2.16), by the associated "gauge potentials" $\mathrm{d}F$ and $\mathrm{d}F_\pi$ as well as by the target space variables ϕ, λ, $\tilde{\chi}^{(+)}$, $\tilde{\bar{\chi}}^{(+)}$, $\hat{\chi}^{(-)}$ and $\hat{\bar{\chi}}^{(-)}$. In the general solution the spinorial gauge potentials are determined by (3.18) and (3.32), $\tilde{A}_C$ is given by (3.34) and π inside the prepotential must be expressed by C_π and spinorial terms according to (3.33).

3.4 Vanishing Casimir C

The other extreme is the case $C \equiv 0$. Then the definition of $\hat{\chi}^{(-)}$ decouples completely from $\bar{\zeta}_{(-)}$. This implies that the latter spinors cannot be expressed in terms of the target space variables, the typical situation one encounters if the target space coordinate is a Casimir function of the Poisson manifold. Indeed, as shown in Section (3.1) a fermionic Casimir function occurs for $C = 0$ in the $N = (1,1)$ case [4], which may simply be identified with $\hat{\chi}^{(-)}$. At least in that limit this result should be reproduced here. Nevertheless it is obvious from (3.20) that $\mathrm{d}\hat{\bar{\chi}}^{(-)} \neq 0$ irrespective of the value of C. The new constant of motion coincides with the (for $C \neq 0$ irrelevant) constant $\lambda_0^{(-)}$:

$$\mathrm{d}\lambda_0^{(-)} = \mathrm{d}(e^{-\frac{i}{2}F_\pi}\hat{\chi}^{(-)}) = 0 \qquad\qquad \mathrm{d}\bar{\lambda}_0^{(-)} = \mathrm{d}(e^{\frac{i}{2}F_\pi}\hat{\bar{\chi}}^{(-)}) = 0 \tag{3.35}$$

On the one hand, this result has the expected property to reduce to the one found in $N = (1,1)$ in the limit where the target space is reduced to this theory. On the other hand, the constant of motion cannot be expressed completely in terms of the target space variables because F_π is related to a gauge field. This pecularity appears in the definition of the "second Casimir" C_π as well. Indeed for $C = 0$ the definition (2.16) is ill defined as there appear inverse powers of C. Of course the combination $C \cdot C_\pi$ is a well defined Casimir, but in the limit $C \to 0$ it makes no sense as C and $C \cdot C_\pi$ are no longer independent. The correct solution is found by looking at the equations of motion including the gauge fields: Indeed they could be integrated in full generality in eq. (3.21); for $C = 0$ we find

$$C_\pi = \pi - \frac{\sigma}{4\sqrt{2}}\tilde{\chi}^{(+)}\tilde{\bar{\chi}}^{(+)} - \frac{i}{2}(e^{\frac{i}{2}F_\pi}\lambda_0^{(-)}\zeta_{(-)} - e^{-\frac{i}{2}F_\pi}\bar{\lambda}_0^{(-)}\bar{\zeta}_{(-)}) \; . \tag{3.36}$$

It should be noted that the last term depends on F_π and $\zeta_{(-)}$, i.e. quantities that are not part of the target space. All remaining gauge potentials follow straightforwardly from the result obtained already above.

Obviously, all problems of finding CD coordinates for $C = 0$ are intimately connected with the divergences at $C = 0$ that show up in C_π. As we work throughout with an explicit basis on the Poisson manifold we should ask whether the characteristics of our solution are generic or a pecularity of our choice of coordinates. By analyzing this the meaning of the (non-)existence of CD coordinates should become more transparent.

The first question concerns the existence of a Casimir function. Indeed, a very simple solution for the elimination of the divergences at $C \to 0$ in (2.16) could be that then a second (commuting) Casimir exists for vanishing fermions only. However, the analysis of this Section showed that there exist for all solutions at least two commuting constants of motion, one related to C the other one related to C_π. Therefore it remains to check, whether new coordinates can be chosen in such a way that the Casimir function C_π remains regular. The problem can be considered in two different versions:

1. One can ask whether such coordinates exist in an entire neighborhood of $C = 0$. As such a region includes points where $C \neq 0$ has non-vanishing body, one can reduce this to the question whether there exist coordinates such that C_π remains regular in the limit $C = 0$. Indeed, such coordinates can be defined for a restricted range of the fermionic fields. This is most easily seen from equation (3.33), where by the rescaling

$$\hat{\chi}^{(-)} = \sqrt{C}\check{\chi}^{(-)} , \qquad \hat{\bar{\chi}}^{(-)} = \sqrt{C}\check{\bar{\chi}}^{(-)} \tag{3.37}$$

all divergences from C_π disappear. However from the definition of $\hat{\chi}$ in eq. (3.2) it is seen that this implies in the limit of $C \to 0$

$$\chi^- = \frac{1}{2\sqrt{2}} \frac{u}{X^{++}} \bar{\chi}^+ , \qquad \bar{\chi}^- = \frac{1}{2\sqrt{2}} \frac{\bar{u}}{X^{++}} \chi^+ . \tag{3.38}$$

Clearly, the general solution from (3.20) with independen $\chi^{(-)}$ and $\chi^{(+)}$ need not respect this constraint for $C = 0$.

2. A weaker requirement would be to find regular coordinates that are valid on the sheet $C = 0$ only. Here a similar problem as in the example of Section 3.1 is encountered. Clearly, the system in eqs. (3.4)-(3.6) cannot be transformed to CD coordinates as the inverse of a spinorial quantity is not defined (remember that the remaining coordinates are already CD).

There is yet another way to illustrate the difference with respect to the solutions with $C \neq 0$: The solution with $C = 0$ is parametrized by $C = 0$, $\mathrm{d}F$, C_π, F_π, ϕ, λ, $\tilde{\chi}^{(+)}$, $\tilde{\bar{\chi}}^{(+)}$, $\lambda_0^{(-)}$, $\bar{\lambda}_0^{(-)}$, $\zeta_{(-)}$, $\bar{\zeta}_{(-)}$. Counting the degrees of freedom it is seen that this configuration space is by one real bosonic and one complex fermionic constant larger than the one found for $C \neq 0$, namely by the integration constant of F_π and one constant from $\zeta_{(-)}$ and $\lambda_0^{(-)}$. Both of them appear in eqs. (3.33) and (3.34) but it was seen there that physics do not depend on the value of these constants, but they can be absorbed by simple redefinitions of other free variables. In the present case the situation is different as $\lambda_0^{(-)}$ determines the value of $\hat{\chi}^{(-)}$ and e.g. labels states with different soul contributions to the charge C_π. To reduce the configuration space of the solution at $C = 0$ to the one of $C \neq 0$ in the present setup one has to choose $\lambda_0^{(-)} = 0$. Comparison with (3.20) for $C = 0$ shows that this condition is exactly (3.38). The remaining constant from F_π automatically disappears once this constraint is imposed. We have argued above that $\lambda_0^{(-)}$ replaces the anti-commuting Casimir function that was found in $N = (1, 1)$ supergravity at $C = 0$. Remarkably enough we now find, that the reduction of the configuration space implies that this constant of motion vanishes.

In summary the solutions for $C = 0$ can be divided into two classes: The first class consists of solutions that exist in an entire neighborhood of $C = 0$ and consequently the configuration space has the same dimension as the one for $C \neq 0$. However, there

exist additional solutions that appear due to the integration constants of F_π and $\lambda_0^{(-)}$. These solutions cannot be extended to the case $C \neq 0$ with non-vanishing body.

3.5 Pure soul Casimir

This case lies in-between the cases 3.3 and 3.4. As eq. (3.20) for non-invertible $C \neq 0$ cannot be solved for $\zeta_{(-)}$ the gauge potential $\tilde{A}_{(-)}$ cannot be expressed in terms of the target-space variables. Therefore, the solution is parametrized by the same quantities as in the case $C = 0$. The discussion of the two classes of solutions still applies and again the class of solutions with $\lambda_0^{(-)} = 0$ can be obtained smoothly from solutions with non-vanishing body of C. Nevertheless it is important to notice that this does no longer imply the constraint (3.38), as $\hat{\chi}^{(-)}$ is at least partially defined through $\bar{\zeta}_{(-)}$.

4 Solution for general models

Our main task is to solve $N = (2,2)$ supergravity with $Z \neq 0$, i.e. the models described in Section 2. Their solutions can be obtained by applying conformal transformations interpreted as target space diffeomorphisms to the solutions at $Z = 0$ of the previous Section. In the present Section the variables of the general model of Section 2 now are underlined (cf. footnote 7). According to the formulas of Section 4 in [9] with $Z = Q'$ the new gauge potentials become

$$\underline{\omega} = \omega + \frac{1}{4}\big((Z+\bar{Z})X^b e_b + \bar{Z}\chi\psi + Z\bar{\chi}\bar{\psi}\big) \ , \qquad \underline{B} = B - \frac{i}{4}(\bar{Z}\chi\psi - Z\bar{\chi}\bar{\psi}) \ , \tag{4.1}$$

$$\underline{e}_a = e^{(Q+\bar{Q})/4} e_a \ , \qquad \underline{\psi}_\alpha = e^{\bar{Q}/4}\psi_\alpha \ , \qquad \underline{\bar{\psi}}_\alpha = e^{Q/4}\bar{\psi}_\alpha \ , \tag{4.2}$$

with the conformal factor Q being an analytic function in $X = \phi + i\pi$. The general solution is obtained by taking these linear combinations of the solution of the simplified model in eqs. (3.23)-(3.30). At the same time the target space variables that parametrize these solutions must be transformed according to

$$\underline{X} = X \ , \qquad \underline{X}^a = e^{-(Q+\bar{Q})/4} X^a \ , \qquad \underline{\chi}^\alpha = e^{-\bar{Q}/4}\chi^\alpha \ , \qquad \underline{\bar{\chi}}^\alpha = e^{-Q/4}\bar{\chi}^\alpha \ . \tag{4.3}$$

The prepotential transforms as $\underline{u} = e^{\bar{Q}/2}u$. The definition of the free functions $\mathrm{d}F$, F_π, $\lambda_0^{(-)}$ and $\zeta_{(-)}$ remain unchanged, but the relations (3.18)-(3.20) must be adjusted due to eq. (4.3). The constant of motion in (3.21) changes in such a way that it coincides with (2.16) in the case of $C \neq 0$ with non-vanishing body. The latter Casimir function is given by (2.15).

The main characteristics of the three classes of solutions as discussed in Sections 3.3, 3.4 and 3.5 remain unchanged. For $C \neq 0$ with non-vanishing body eq. (3.20) can

be solved for $\underline{\hat{\chi}}^{(-)}$. For $C = 0$ now

$$\lambda_0^{(-)} = e^{-\frac{i}{2}F_\pi + \frac{1}{4}\bar{Q}} \big(e^{\frac{1}{4}(Q+\bar{Q})} \underline{\tilde{\chi}}^{(-)} - \frac{i\sigma\bar{u}}{2\sqrt{2}} \underline{\tilde{\bar{\chi}}}^{(+)} \big) \tag{4.4}$$

is the anti-commuting constant of motion.

Finally it should be mentioned that all the models considered so far were related to chiral gauging. Twisted chiral gaugings are obtained [9] by the change of variables $\chi^- \leftrightarrow \bar{\chi}^-$ and $\psi_- \leftrightarrow \bar{\psi}_-$. As this redefinition is defined globally, the discussion of the twisted chiral case is completely analogous to the one of chiral gauging.

5 Ungauged supergravity

Beside the two versions of minimally gauged $N = (2, 2)$ supergravity discussed so far ungauged versions have been found in the context of superstring compactifications [20–22]. It was shown by us in [9] that such models can be obtained from the Poisson tensor (2.8)-(2.14) by decoupling the scalar field π and its associated gauge field B. This is done by a change of variables; instead of π the Casimir function C_π is used as a new target space coordinate. Then, as $\{C_\pi, X^I\} \equiv 0$ for all fields X^I, the corresponding part of the Poisson tensor can be dropped and $u(X, \bar{X})$ and $Q(X)$ become functions of the dilaton and the dilatinos[8].

Again we restrict the explicit calculations to dilaton prepotential supergravity $Z = 0$. As π appears in the prepotential $u(\phi + i\pi)$ and $\bar{u}(\phi - i\pi)$ the relevant replacement is

$$\begin{aligned} u(\phi + i\pi) = \hat{u} &+ \frac{1}{4C_B}\hat{u}'\big(\hat{\bar{u}}\chi^2 - \hat{u}\bar{\chi}^2 + 4iX^a(\chi\gamma_a\gamma_*\bar{\chi})\big) \\ &+ \frac{1}{16C_B}\chi^2\bar{\chi}^2\big(\hat{u}'' + \frac{1}{C_B}(\hat{u}\hat{\bar{u}}' - \hat{u}'\hat{\bar{u}})\big) \ . \end{aligned} \tag{5.1}$$

Here $\hat{u}(\phi + iC_\pi)$ is the prepotential after the replacement $\pi \to C_\pi$ and $C_B = 8Y - \hat{u}\hat{\bar{u}}$ is the body of the Casimir function C with respect to the ungauged model.

To determine the solution of the ungauged model we could start from the explicit expansion of the prepotential in terms of the Casimir C_π in (5.1). Then we could determine the solution in terms of the new coordinates

$$\check{X}^I = (C, C_\pi, \phi, \lambda, \tilde{\chi}^{(+)}, \tilde{\bar{\chi}}^{(+)}, \hat{\chi}^{(-)}, \hat{\bar{\chi}}^{(-)}) \ . \tag{5.2}$$

This should reproduce the solution of Section 3.3 and, after dropping the Casimir C_π, lead to the solution for the ungauged model as well. However, the calculation of the

[8] In a more mathematical language, a fixed symplectic leaf with respect to the foliation by C_π is chosen. Thus the ungauged model has a smaller configuration space than the gauged model, which includes all symplectic leaves.

corresponding brackets is very complicated as one has to expand the prepotential in (3.2) as well.

Fortunately, there exists a simple trick to obtain the solution in a straightforward way. We can view the replacement $\pi \to C_\pi$ as a target space diffeomorphism and simply apply the ensuing transformation rules of the gauge fields to the solution obtained in Section 3.2. From the expansion of C_π in terms of the variables $\tilde{X}^I$ (cf. the paragraph below eq. (3.23))

$$C_\pi = \pi - \frac{\sigma}{4\sqrt{2}}\tilde{\chi}^{(+)}\tilde{\tilde{\chi}}^{(+)} - \frac{\sqrt{2}\sigma}{C}\hat{\chi}^{(-)}\hat{\hat{\chi}}^{(-)} \tag{5.3}$$

and the solution of Section 3.3 one finds

$$\check{A}_{C_\pi} = \tilde{A}_\pi = -\,\mathrm{d}F_\pi \ , \qquad \check{A}_\lambda = \mathrm{d}\phi \ , \qquad \check{A}_\phi = -\,\mathrm{d}\lambda \ , \tag{5.4}$$

$$\check{A}_C = \tilde{A}_C + \frac{\sqrt{2}\sigma}{C^2}\hat{\chi}^{(-)}\hat{\hat{\chi}}^{(-)}\,\mathrm{d}F_\pi = -\,\mathrm{d}F + \frac{\sqrt{2}i\sigma}{C^2}(\hat{\hat{\chi}}^{(-)}\,\mathrm{d}\hat{\chi}^{(-)} + \hat{\chi}^{(-)}\,\mathrm{d}\hat{\hat{\chi}}^{(-)}) \ , \tag{5.5}$$

$$\check{A}_{(+)} = \tilde{A}_{(+)} - \frac{\sigma}{4\sqrt{2}}\tilde{\tilde{\chi}}^{(+)}\,\mathrm{d}F_\pi = \frac{i\sigma}{2\sqrt{2}}\,\mathrm{d}\tilde{\tilde{\chi}}^{(+)} \ , \tag{5.6}$$

$$\check{\tilde{A}}_{(+)} = \tilde{\tilde{A}}_{(+)} + \frac{\sigma}{4\sqrt{2}}\tilde{\chi}^{(+)}\,\mathrm{d}F_\pi = \frac{i\sigma}{2\sqrt{2}}\,\mathrm{d}\tilde{\chi}^{(+)} \ , \tag{5.7}$$

$$\check{A}_{(-)} = \tilde{A}_{(-)} - \frac{\sqrt{2}\sigma}{C}\hat{\hat{\chi}}^{(-)}\,\mathrm{d}F_\pi = i\frac{2\sqrt{2}i\sigma}{C}\,\mathrm{d}\hat{\hat{\chi}}^{(-)} \ , \tag{5.8}$$

$$\check{\tilde{A}}_{(-)} = \tilde{\tilde{A}}_{(-)} + \frac{\sqrt{2}\sigma}{C}\hat{\chi}^{(-)}\,\mathrm{d}F_\pi = -\frac{2\sqrt{2}i\sigma}{C}\,\mathrm{d}\hat{\chi}^{(-)} \ . \tag{5.9}$$

As expected $\mathrm{d}F_\pi$ does no longer appear in the transformed expressions—except in the first equation of (5.4) of course—and thus C_π may be eliminated consistently.

To transform this solution to the defining coordinates of the ungauged model the prepotential in (3.2) must be replaced by the expansion (5.1):

$$\hat{\chi}^{(-)} = \tilde{\chi}^{(-)} - \frac{i\sigma\hat{u}}{2\sqrt{2}}\tilde{\tilde{\chi}}^{(+)} - \frac{i\sigma}{4\sqrt{2}C_B}\hat{u}'\hat{\hat{u}}\tilde{\chi}^{(-)}\tilde{\chi}^{(+)}\tilde{\tilde{\chi}}^{(+)} + \frac{1}{2C_B}\hat{u}'\tilde{\chi}^{(-)}\tilde{\tilde{\chi}}^{(-)}\tilde{\tilde{\chi}}^{(+)} \tag{5.10}$$

Now the original gauge fields of the ungauged models can be obtained by the transformation rules of the target space diffeomorphisms. Notice that C_B is a function of Y and ϕ, derivatives must be taken with respect to these variables as well. Also, the Lorentz invariant combinations for the spinors must be expressed again in terms of the original fields.

It was pointed out already in ref. [9] that the ungauged model allows for a restricted class of solutions with $C = 0$ only. This can be made more explicit at this point: The solutions of the ungauged model correspond to those of the gauged one where C_π can be expressed in terms of target-space coordinates alone (cf. the discussion in Section

3.4). But it was found in the previous Section that these are exactly the solutions with $\lambda_0^{(-)} = 0$. Therefore for the ungauged model the configuration space at $C = 0$ is the same as for $C \neq 0$, the additional solutions found in the minimally gauged model disappear. The ensuing restriction can be made manifest from eq. (5.10). For $C = 0$ one obtains the condition $\hat{\chi}^{(-)} = 0$, for C pure soul this variable is related to the Casimir by (3.20). Though these relations have the same origin as in the minimally gauged model it should be realized that the solution in terms of the physical coordinates are different, as the prepotential must be expanded in terms of C_π in the present case.

6 BPS solutions

Solutions that preserve some of the supersymmetries play an important rôle in many different aspects of supergravity theories. In [25] it was shown for $N = (1,1)$ that the gPSM approach to dilaton supergravity is very powerful in the discussion of BPS states as well. In this Section we present first steps of an extension to minimally gauged $N = (2,2)$ supergravity. Beside technical complications the main difference is the appearance of new bosonic fields and of an additional bosonic Casimir function.

In contrast to [25] our present discussion is restricted to bosonic field configurations only. Such a configuration is BPS if the supersymmetry variations of the fermionic variables vanish. From (2.5) these transformations in this simplified case are:

$$\delta\chi^+ = -2\sqrt{2}X^{++}\bar{\varepsilon}_+ - u\varepsilon_- \tag{6.1}$$

$$\delta\chi^- = -2\sqrt{2}X^{--}\bar{\varepsilon}_- - u\varepsilon_+ \tag{6.2}$$

$$\delta\psi_+ = -D\varepsilon_+ + \sqrt{2}e^{-Q/2}\bar{w}'\bar{\varepsilon}_-e_{++} + \frac{\bar{Z}}{2}X^{--}e_{--}\varepsilon_+ \tag{6.3}$$

$$\delta\psi_- = -D\varepsilon_- - \sqrt{2}e^{-Q/2}\bar{w}'\bar{\varepsilon}_+e_{--} - \frac{\bar{Z}}{2}X^{++}e_{++}\varepsilon_- \tag{6.4}$$

6.1 Full supersymmetry

States that respect all supersymmetries must have $X^a = 0$. Furthermore the complex dilaton $X = \phi + i\pi$ must be chosen such that $u(X_{BPS}, \bar{X}_{BPS}) = 0$ and $u'(X_{BPS}, \bar{X}_{BPS}) = 0$. Solutions of this type are invariant under all supersymmetries if the transformations parameters are covariantly constant. The Casimir function C in (2.15) vanishes for this solution. The requirement that the fully supersymmetric state is a ground state fixes the additive ambiguity in the definition (2.15). The Casimir related to the $U(1)$ is not restricted to a specific value. Its possible values depend on the details of the prepotential and are determined by the condition $u = u' = 0$.

The eom of the complex dilaton reduces to $\mathrm{d}X = 0$ and therefore the solutions belong to the special class of "constant dilaton vacua" CDV [25,33]. As $u' = 0$ the eom of the spin connection reduces to $\mathrm{d}\omega = 0$ and thus curvature vanishes.

6.2 BPS states

In many applications the interesting field configurations are restricted to vanishing fermion fields. Eqs. (6.1) and (6.2) imply

$$u\varepsilon_- = -2\sqrt{2}X^{++}\bar{\varepsilon}_+ \ , \qquad u\varepsilon_+ = -2\sqrt{2}X^{--}\bar{\varepsilon}_- \ . \tag{6.5}$$

Iteration of these equations (and their hermitian conjugates) imply that $C = 0$. This is equivalent to the statement that the determinant of the purely fermionic part of the Poisson tensor must vanish. There exist three different types of solutions.

6.2.1 CDV solutions

If $X^{++} = X^{--} = 0$ the complex dilaton X is again constant and the prepotential vanishes on the solution. However $u' \neq 0$, else the fully supersymmetric solution would be recovered. from the eom for the spin connection one deduces

$$R = 2 * \mathrm{d}\omega = \frac{1}{2}u'\bar{u}' > 0 \ , \tag{6.6}$$

which in our conventions implies AdS space. The solutions do not respect full supersymmetry as from (6.3) and (6.4)

$$D\varepsilon_+ - \frac{1}{2\sqrt{2}}\bar{u}'e_{++}\bar{\varepsilon}_- = 0 \ , \qquad D\varepsilon_- + \frac{1}{2\sqrt{2}}\bar{u}'e_{--}\bar{\varepsilon}_+ = 0 \ . \tag{6.7}$$

6.2.2 Chiral solutions

Even with the choice $X^{++} \neq 0$ it may happen that $X^{--} = u = 0$. In that case $\varepsilon_+ = 0$ and only the ε_- component can be non-zero. As for all solutions with vanishing fermions $\mathrm{d}\pi = 0$, but now $\mathrm{d}\phi \neq 0$. Together with the condition $u = 0$ on the solution this however implies $u \equiv 0$. Therefore all chiral solutions are flat Minkowski space. As $X^{++} \neq 0$ this case is covered by the discussion of Section 4. Integration of eq. (6.4) yields

$$\varepsilon_- = \exp\Big[\frac{1}{2}(\bar{Q} - iF_\pi)\Big]\sqrt{|X^{++}|}\tilde{\varepsilon} \tag{6.8}$$

with a constant spinor $\tilde{\varepsilon}$. All states of this type respect two of the four supersymmetries.

6.2.3 Supersymmtric black holes

Obviously the cases 6.2.1 and 6.2.2 do not describe supersymmetric black hole solutions. The latter only exist if all three quantities X^{++}, X^{--} and u are different from zero. Then from (6.5) it follows that both components ε_+ and ε_- must be nonvanishing. It remains to check the differential equations (6.3) and (6.4). By the use of the explicit solution derived in the previous Sections it can be shown that (6.4) is the hermitian conjugate of (6.3). Thus it remains to solve a single differential equation that can be written as

$$\mathrm{d}\varepsilon_- = \mathrm{d}\big(\frac{1}{2}\ln X^{++} - \frac{i}{2}F_\pi + \frac{1}{2}\bar{Q}\big)\varepsilon_- + 2(Z - \bar{Z})W\,\mathrm{d}F\varepsilon_- \ . \tag{6.9}$$

If Z is real the solution for ε_- is given by eq. (6.8) while ε_+ reads:

$$\varepsilon_+ = -\frac{i\sigma}{2\sqrt{2}}\exp\big[\frac{1}{2}(\bar{Q} - iF_\pi)\big]\frac{u}{\sqrt{|X^{++}|}}\tilde{\varepsilon} \tag{6.10}$$

In the general case a closed analytic expression cannot be obtained. Again all solutions respect half of the supersymmetries.

As a result of this Section it follows that any bosonic field configuration with $C = 0$ *locally* is BPS. At the same time it should be realized that this need not be true globally. Indeed, global solutions in the general case are obtained by a combination of several patches, which may destroy the BPS property at the global level (cf. [34]). Finally we mention the agreement of these calculations with general statements on supersymmetric black hole solutions [35–37]: all supersymmetric black holes are extremal. In our calculations this immediately follows from the Killing norm for $C = 0$ [25, 27]

$$K(X) = -2e^{(Q+\bar{Q})/2}W = \frac{1}{4}\big|e^{\bar{Q}}u\big|^2 \ . \tag{6.11}$$

Obviously all zeros are of even degree.

7 Conclusions

The present work on $N = (2, 2)$ supergravity in two dimensions extends our previous one [9] by providing for the first time the full classical solutions, including the complete fermionic parts. This is possible thanks to the powerful tool of the equivalent formulation as a particular class of graded Poisson Sigma models. Although the actual computation is restricted to the chiral case, the twisted chiral $N = (2, 2)$ theories, as well as the ungauged version can be obtained by simple redefinitions in the formulas presented here. The classification of the solutions is determined by the values of the Casimir functions in which the interplay between body and soul characterizes different

cases. Although we draw heavily from our experience with the $N = (1, 1)$ case [4,6], the present results exhibit new interesting structures due to the larger fermionic symmetry algebra.

As yet another application of research directions which are possible in the framework of gPSMs we discuss solutions retaining certain supersymmetries (BPS states), although in $N = (2, 2)$ supergravity the analysis turns out to be rather more involved than the one in $N = (1, 1)$. Therefore, only bosonic solutions are treated here.

Comparing with other results, already obtained for $N = (1, 1)$ it is clear that beside a more comprehensive discussion of BPS states in analogy to ref. [25], also the problem of putting a (super) point particle into an $N = (2, 2)$ background (cf. [6] for $N = (1, 1)$), the coupling of supersymmetric matter (cf. [25]) as well as the quantization of $N = (2, 2)$ supergravity (cf. [23, 24]) are topics expected to allow a successful treatment in further work. This will allow new insights also for application in superstring theory where the gPSM approach now seems to provide a new line of attack for the solution of some old problems.

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A Notations and conventions

The conventions are identical to [4, 28], where additional explanations can be found.

Indices chosen from the Latin alphabet are generic (upper case) or (lower case) refer to commuting objects, Greek indices are anti-commuting ones. Holonomic coordinates are labeled by M, N, O etc., anholonomic ones by A, B, C etc., whereas I, J, K etc. are general indices of the gPSM:

$$X^I = (X^\phi, X^\pi, X^a, X^\alpha, X^{\bar{\alpha}}) = (\phi, \pi, X^a, \chi^\alpha, \bar{\chi}^\alpha) \tag{A.1}$$

$$A_I = (A_\phi, A_\pi, A_a, A_\alpha, A_{\bar{\alpha}}) = (\omega, B, e_a, \psi_\alpha, \bar{\psi}_\alpha) \tag{A.2}$$

The summation convention is always $NW \to SE$, e.g. for a fermion χ: $\chi^2 = \chi^\alpha \chi_\alpha$. Our conventions are arranged in such a way that almost every bosonic expression is transformed trivially to the graded case when using this summation convention and replacing commuting indices by general ones. This is possible together with exterior derivatives acting *from the right*, only. Thus the graded Leibniz rule is given by

$$\mathrm{d}\,(AB) = A\mathrm{d}B + (-1)^B\,(\mathrm{d}A)B\ . \tag{A.3}$$

In terms of anholonomic indices the metric and the symplectic 2×2 tensor are defined as

$$\eta_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \quad \epsilon_{ab} = -\epsilon^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ , \quad \epsilon_{\alpha\beta} = \epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ . \tag{A.4}$$

The metric in terms of holonomic indices is obtained by $g_{mn} = e_n^b e_m^a \eta_{ab}$ and for the determinant the standard expression $e = \det e_m^a = \sqrt{-\det g_{mn}}$ is used. The volume form reads $\epsilon = \frac{1}{2}\epsilon^{ab} e_b \wedge e_a$; by definition $*\epsilon = 1$.

The γ-matrices are used in a chiral representation:

$$\gamma^0{}_\alpha{}^\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^1{}_\alpha{}^\beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \gamma_{*\alpha}{}^\beta = (\gamma^1\gamma^0)_\alpha{}^\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.5}$$

Covariant derivatives of anholonomic indices with respect to the geometric variables $e_a = \mathrm{d}x^m e_{am}$ and $\psi_\alpha = \mathrm{d}x^m \psi_{\alpha m}$ include the two-dimensional spin-connection one form $\omega^{ab} = \omega\epsilon^{ab}$. When acting on lower indices the explicit expressions read ($\frac{1}{2}\gamma_*$ is the generator of Lorentz transformations in spinor space):

$$(De)_a = \mathrm{d}e_a + \omega\epsilon_a{}^b e_b \qquad\qquad (D\psi)_\alpha = \mathrm{d}\psi_\alpha - \frac{1}{2}\omega\gamma_{*\alpha}{}^\beta\psi_\beta \tag{A.6}$$

Dirac conjugation is defined as $\bar\chi^\alpha = \chi^\dagger\gamma_0$. Written in components of the chiral representation

$$\chi^\alpha = (\chi^+, \chi^-) \ , \qquad\qquad \chi_\alpha = \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} \tag{A.7}$$

the relation between upper and lower indices becomes $\chi^+ = \chi_-$, $\chi^- = -\chi_+$. Dirac conjugation follows as $\bar\chi_- = \chi_-^*$, $\bar\chi_+ = -\chi_+^*$, i.e. for Majorana spinors χ_- is real while χ_+ is imaginary.

For two gauge-covariant Dirac spinors χ_α and λ_α the combinations

$$\chi\lambda \ , \qquad \chi\gamma_*\lambda \ , \qquad \bar\chi\gamma^a\lambda \tag{A.8}$$

and their hermitian conjugates are gauge invariant for chiral gaugings, while

$$\bar\chi\lambda \ , \qquad \bar\chi\gamma_*\lambda \ , \qquad \bar\chi\gamma^a\lambda \tag{A.9}$$

are invariant for twisted-chiral gaugings. Note that in the latter case the gravitino ψ_α transforms under gauge transformations as $\bar\chi_\alpha$. Thus in eq. (A.9) the bilinear invariants of a gravitino and a dilatino are obtained by substituting $\lambda \to \bar\psi$.

Vectors in light-cone coordinates are given by

$$v^{++} = \frac{i}{\sqrt{2}}(v^0 + v^1) \ , \qquad\qquad v^{--} = \frac{-i}{\sqrt{2}}(v^0 - v^1) \ . \tag{A.10}$$

The additional factor i in (A.10) permits a direct identification of the light-cone components with the components of the spin-tensor $v^{\alpha\beta} = \frac{i}{\sqrt{2}} v^c \gamma_c^{\alpha\beta}$. This implies that $\eta_{++|--} = 1$ and $\epsilon_{--|++} = -\epsilon_{++|--} = 1$. The γ-matrices in light-cone coordinates become

$$(\gamma^{++})_\alpha{}^\beta = \sqrt{2} i \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \qquad (\gamma^{--})_\alpha{}^\beta = -\sqrt{2} i \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} . \tag{A.11}$$

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